

# Portfolio Optimisation with Threshold Accepting

Enrico Schumann  
es@enricoschumann.net

This vignette provides the code for some of the examples from Gilli et al. [2011]. For more details, please see Chapter 13 of the book; the code in this vignette uses the scripts `exampleSquaredRets.R`, `exampleSquaredRets2.R` and `exampleRatio.R`.

We start by attaching the package. We will later on need the function `resample` (see `?sample`).

```
> require("NMOF")
> resample <- function(x, ...)
  x[sample.int(length(x), ...)]
> set.seed(112233)
```

## 1 Minimising squares

### 1.1 A first implementation

This problem serves as a benchmark: we wish to find a long-only portfolio  $w$  (weights) that minimises squared returns across all return scenarios. These scenarios are stored in a matrix  $R$  of size number of scenarios  $ns$  times number of assets  $na$ . More formally, we want to solve the following problem:

$$\begin{aligned} \min_w \Phi & & (1) \\ w^t \mathbf{1} &= 1, \\ 0 \leq w_j &\leq w_j^{\text{sup}} \quad \text{for } j = 1, 2, \dots, n_A. \end{aligned}$$

We set  $w_j^{\text{sup}}$  to 5% for all assets.  $\Phi$  is the squared return of the portfolio,  $w'R'Rw$ , which is similar to the portfolio return's variance. We have

$$\frac{1}{n_S} R'R = \text{Cov}(R) + mm'$$

in which  $\text{Cov}$  is the variance–covariance matrix operator, which maps the columns of  $R$  into their variance–covariance matrix;  $m$  is a column vector that holds the column means of  $R$ , ie,  $m' = \frac{1}{n_S} \mathbf{1}'R$ . For short time horizons, the mean of a column is small compared with the average squared return of the column. Hence, we ignore the matrix  $mm'$ , and variance and squared returns become equivalent.

For testing purposes we use the matrix `fundData` for  $R$ .

```
> na <- dim(fundData)[2L]
> ns <- dim(fundData)[1L]
> winf <- 0.0; wsup <- 0.05
> data <- list(R = t(fundData),
  RR = crossprod(fundData),
  na = na,
  ns = ns,
  eps = 0.5/100,
  winf = winf,
  wsup = wsup,
  resample = resample)
```

The neighbourhood function automatically enforces the budget constraint.

```
> neighbour <- function(w, data){
  eps <- runif(1L) * data$eps
  toSell <- w > data$winf
```

```

toBuy <- w < data$wsup
i <- data$resample(which(toSell), size = 1L)
j <- data$resample(which(toBuy), size = 1L)
eps <- min(w[i] - data$winf, data$wsup - w[j], eps)
w[i] <- w[i] - eps
w[j] <- w[j] + eps
w
}

```

The objective function.

```

> OF1 <- function(w, data) {
  Rw <- crossprod(data$R, w)
  crossprod(Rw)
}
> OF2 <- function(w, data) {
  aux <- crossprod(data$RR, w)
  crossprod(w, aux)
}

```

OF2 uses  $R'R$ ; thus, it does not depend on the number of scenarios. But this is only possible for this very specific problem.

We specify a random initial solution  $w_0$  and define all settings in a list `algo`.

```

> w0 <- runif(na); w0 <- w0/sum(w0)
> algo <- list(x0 = w0,
  neighbour = neighbour,
  nS = 2000L,
  nT = 10L,
  nD = 5000L,
  q = 0.20,
  printBar = FALSE,
  printDetail = FALSE)

```

We can now run `TAopt`, first with `OF1` ...

```
> system.time(res <- TAopt(OF1,algo,data))
```

```

  user  system elapsed
21.547  23.035   3.973

```

```
> 100 * sqrt(crossprod(fundData %*% res$xbest)/ns)
```

```

  [,1]
[1,] 0.33635

```

... and then with `OF2`.

```
> system.time(res <- TAopt(OF2,algo,data))
```

```

  user  system elapsed
15.061  16.016   2.605

```

```
> 100*sqrt(crossprod(fundData %*% res$xbest)/ns)
```

```

  [,1]
[1,] 0.33655

```

Note that we have rescaled the results (see the book for details). Both results are similar, but `OF2` typically requires less time. We check the constraints.

```
> min(res$xbest) ## should not be smaller than data$winf
```

```
[1] 0
```

```
> max(res$xbest) ## should not be greater than data$wsup
```

```
[1] 0.05
```

```
> sum(res$xbest) ## should be one
```

```
[1] 1
```

The problem can actually be solved quadratic programming; we use the quadprog package [Turlach and Weingessel, 2019].

```
> if (require("quadprog", quietly = TRUE)) {
  covMatrix <- crossprod(fundData)
  A <- rep(1, na); a <- 1
  B <- rbind(-diag(na), diag(na))
  b <- rbind(array(-data$wsup, dim = c(na, 1L)),
             array( data$winf, dim = c(na, 1L)))
  system.time({
    result <- solve.QP(Dmat = covMatrix,
                      dvec = rep(0,na),
                      Amat = t(rbind(A,B)),
                      bvec = rbind(a,b),
                      meq = 1L)
  })
  wqp <- result$solution

  cat("Compare results...\n")
  cat("QP:", 100 * sqrt( crossprod(fundData %*% wqp)/ns ), "\n")
  cat("TA:", 100 * sqrt( crossprod(fundData %*% res$xbest)/ns ) , "\n")

  cat("Check constraints ...\n")
  cat("min weight:", min(wqp), "\n")
  cat("max weight:", max(wqp), "\n")
  cat("sum of weights:", sum(wqp), "\n")
}
```

```
Compare results...
QP: 0.33612
TA: 0.33655
Check constraints ...
min weight: -9.748e-17
max weight: 0.05
sum of weights: 1
```

## 1.2 Updating

Here we implement the updating of the objective function as described in Gilli et al. [2011].

```
> neighbourU <- function(sol, data){
  wn <- sol$w
  toSell <- wn > data$winf
  toBuy <- wn < data$wsup
  i <- data$resample(which(toSell), size = 1L)
  j <- data$resample(which(toBuy), size = 1L)
  eps <- runif(1) * data$eps
  eps <- min(wn[i] - data$winf, data$wsup - wn[j], eps)
  wn[i] <- wn[i] - eps
  wn[j] <- wn[j] + eps
  Rw <- sol$Rw + data$R[,c(i,j)] %*% c(-eps,eps)
  list(w = wn, Rw = Rw)
```

```

}
> OF <- function(sol, data)
  crossprod(sol$Rw)

```

Prepare the data list (we reuse several items used before).

```

> data <- list(R = fundData, na = na, ns = ns,
  eps = 0.5/100, winf = winf, wsup = wsup,
  resample = resample)

```

We start, again, with a random solution, and also use the same number of iterations as before.

```

> w0 <- runif(data$na); w0 <- w0/sum(w0)
> x0 <- list(w = w0, Rw = fundData %*% w0)
> algo <- list(x0 = x0,
  neighbour = neighbourU,
  nS = 2000L,
  nT = 10L,
  nD = 5000L,
  q = 0.20,
  printBar = FALSE,
  printDetail = FALSE)
> system.time(res2 <- TAOpt(OF, algo, data))

```

```

  user  system elapsed
1.173   0.561   0.817

```

```

> 100*sqrt(crossprod(fundData %*% res2$xbest$w)/ns)

```

```

  [,1]
[1,] 0.33653

```

This should be faster, and we arrive at the same solution as before.

### 1.3 Redundant assets

We duplicate the last column of fundData.

```

> fundData <- cbind(fundData, fundData[, 200L])

```

Thus, while the dimension increases, the column rank stays unchanged.

```

> dim(fundData)

```

```

[1] 500 201

```

```

> qr(fundData)$rank

```

```

[1] 200

```

```

> qr(cov(fundData))$rank

```

```

[1] 200

```

Checking the weight of the last asset (which was zero), we know that the solution to our model must be unchanged, too.

```

> if (require("quadprog", quietly = TRUE))
  wqp[200L]

```

```

[1] 2.0755e-17

```

We redo our example.

```

> na <- dim(fundData)[2L]
> ns <- dim(fundData)[1L]
> winf <- 0.0; wsup <- 0.05
> data <- list(R = fundData, na = na, ns = ns,
              eps = 0.5/100, winf = winf, wsup = wsup,
              resample = resample)
>

```

But a number of QP solvers have problems with such cases.

```

> if (require("quadprog", quietly = TRUE)) {
  covMatrix <- crossprod(fundData)
  A <- rep(1, na); a <- 1
  B <- rbind(-diag(na), diag(na))
  b <- rbind(array(-data$wsup, dim = c(na, 1L)),
             array( data$winf, dim = c(na, 1L)))
  cat(try(result <- solve.QP(Dmat = covMatrix,
                           dvec = rep(0,na),
                           Amat = t(rbind(A,B)),
                           bvec = rbind(a,b),
                           meq = 1L)
      ))
}

```

```

Error in solve.QP(Dmat = covMatrix, dvec = rep(0, na), Amat = t(rbind(A,
matrix D in quadratic function is not positive definite!

```

But TA can handle this case.

```

> w0 <- runif(data$na); w0 <- w0/sum(w0)
> x0 <- list(w = w0, Rw = fundData %*% w0)
> algo <- list(x0 = x0,
              neighbour = neighbourU,
              nS = 2000L,
              nT = 10L,
              nD = 5000L,
              q = 0.20,
              printBar = FALSE,
              printDetail = FALSE)
> system.time(res3 <- TAOpt(OF, algo, data))

```

```

user system elapsed
1.136 0.555 0.753

```

```

> 100*sqrt(crossprod(fundData %*% res3$xbest$w)/ns)

```

```

[,1]
[1,] 0.33671

```

Final check: weights for asset 200 and its twin, asset 201.

```

> res3$xbest$w[200:201]

```

```

[1] 0 0

```

See Gilli et al. [2011, Section 13.2.5] for a discussion of rank-deficiency and its (computational and empirical) consequences for such problems.

## References

Manfred Gilli, Dietmar Maringer, and Enrico Schumann. *Numerical Methods and Optimization in Finance*. Elsevier/Academic Press, 2011. URL <http://enricoschumann.net/NMOF>.

Berwin A. Turlach and Andreas Weingessel. *quadprog: Functions to Solve Quadratic Programming Problems*, 2019. URL <https://CRAN.R-project.org/package=quadprog>. R package version 1.5-7 (S original by Berwin A. Turlach; R port by Andreas Weingessel).